

# Multiplicity fluctuations in relativistic gases.

## From simple models to experiment.

Viktor Begun<sup>1</sup>

<sup>1</sup>*Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine.*

The aim of this paper is to give a short overview for the set of publications considering recently found effect of non-equivalence of multiplicity fluctuations in relativistic gases with globally conserved charge and energy.

### I. INTRODUCTION

It was suggested to use the statistical approach to strong interactions more than 50 years ago [1], [2], [3]. It appeared to be surprisingly successful in describing experimental results on hadron production properties in nuclear collisions at high energies (see e.g. Ref. [4], [5], [6] and references therein). This motivates a rapid development of statistical models and it raises new questions, previously not addressed in statistical physics. In particular, an applicability of the models formulated within various statistical ensembles. Recently, it was found that global conservation laws suppress multiplicity fluctuations and this suppression survive even in thermodynamic limit [7], [8]. This unexpected result gave rise to the set of publications on this subject [7]-[22]. This paper gives a short overview starting from simple models [7], [8] to the recently found experimental confirmation of this effect [19], [20].

### II. MULTIPLICITY FLUCTUATIONS

Multiplicity fluctuations can be quantified by the scaled variance. For positively, and negatively, charged particles the scaled variance reads:

$$\omega^{\pm} \equiv \frac{\langle N_{\pm}^2 \rangle - \langle N_{\pm} \rangle^2}{\langle N_{\pm} \rangle}, \quad (1)$$

where angular brackets  $\langle \rangle$  means averaging. The scaled variance is a useful measure, because for Poisson distribution it equals 1, independently of its mean value:

$$\omega_{poisson}^{\pm} = 1 \quad (2)$$

Thus, the scaled variance says how much the studied system is different from Poisson distribution. Experimentally, the averaging in the Eq. (1) means the averaging on event-by-event

basis: a given observable is measured in each collision event and the fluctuations are evaluated for the selected set of these events (see, e.g., review [6]). To calculate a statistical "background" for multiplicity fluctuations one has to choose a statistical ensemble for this calculation: grand canonical (GCE), canonical (CE), microcanonical (MCE) or grand microcanonical (GMCE), see Fig. 1. Usually authors do not make the difference between MCE

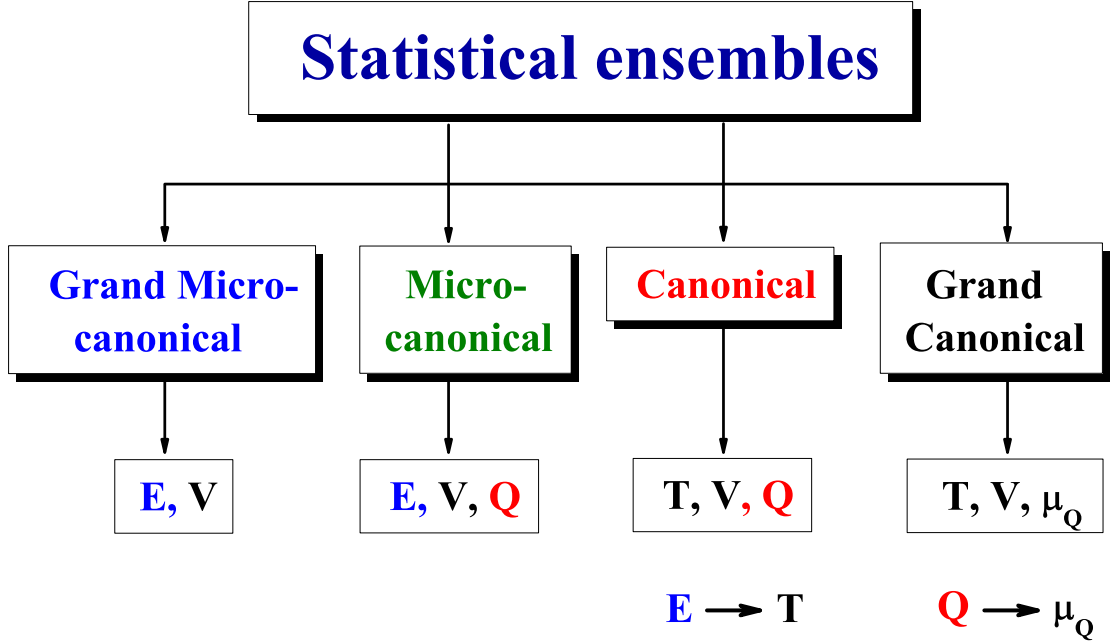


FIG. 1: Conservation laws in different statistical ensembles.

and GMCE and call both as microcanonical ensemble. We introduce different names following the suggestion of a referee, because we analyze and compare different ensembles in details.

The choice of an ensemble depends on the experimental situation. If one exactly knows the energy, volume and charge of the system then such a system should be described in the MCE. Sometimes temperature of a system with exactly known electric charge can be measured much easier than its whole energy. Then such a system should be treated in CE, etc... In practice, calculations in CE and especially in GMCE and MCE are very difficult thus real calculations are always performed in GCE. One usually refers here to the textbook statement that all ensembles are equivalent in thermodynamic limit.

This is the case for particle multiplicities. Different ensembles are equivalent if one choose a temperature and chemical potentials in a way that some exactly fixed variable in one ensemble equals to its adjoint average value in another ensemble, e.g. temperature  $T$  is defined from the condition  $E_{m.c.e.} = \langle E \rangle_{c.e.}$ , and chemical potential  $\mu_Q$  from the condition  $Q_{c.e.} = \langle Q \rangle_{g.c.e.}$ , etc..., see Fig. 1. However the equivalence of statistical ensembles does not

apply to scaled variances. This was firstly found in [7] and will be illustrated below.

### A. Canonical ensemble

As a simplest example, let us consider a relativistic system in equilibrium which consists of one sort of positively,  $N_+$ , and negatively charged particles<sup>1</sup>,  $N_-$ , with total charge equal to  $Q_{c.e.} = N_+ - N_-$ . In the case of the Boltzmann ideal gas (the interactions and quantum statistics effects are neglected) in the volume  $V$  and at temperature  $T$  the GCE and CE partition functions read:

$$\begin{aligned} Z_{g.c.e.}(T, V, \mu_Q) &= \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!} \frac{(\lambda_- z)^{N_-}}{N_-!} e^{\mu_Q(N_+ - N_-)/T} \\ &= \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} Z_{N_+, N_-}(T, V, \mu_Q) = \exp(2z \cosh[\mu_Q/T]), \end{aligned} \quad (3)$$

$$\begin{aligned} Z_{c.e.}(T, V, Q) &= \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} \frac{(\lambda_+ z)^{N_+}}{N_+!} \frac{(\lambda_- z)^{N_-}}{N_-!} \delta(Q - [N_+ - N_-]) \\ &= \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} Z_{N_+, N_-}(T, V, Q) \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\phi \exp[iQ\phi + z(\lambda_+ e^{i\phi} + \lambda_- e^{-i\phi})] = I_Q(2z), \end{aligned} \quad (4)$$

where  $z$  is a single particle partition function:

$$z = \frac{gV}{2\pi^2} \int_0^{\infty} p^2 dp e^{-\frac{\sqrt{p^2 + m^2}}{T}} = \langle N_{\pm} \rangle, \quad (5)$$

$g$  is a degeneracy factor (number of spin states),  $m$  - particle mass and  $\lambda_{\pm}$  are auxiliary parameters that will be set to unity after calculation of average values. We also labelled the number of particles in GCE as  $\langle N_{\pm} \rangle$ . Let us omit the indexes *c.e.*, *g.c.e.*, etc., for partition function as the arguments of  $Z$  already show to what ensemble it corresponds. The average values in both the GCE and CE can be calculated as follows:

$$\langle N_{\pm} \rangle \equiv \frac{1}{Z} \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} N_{\pm} Z_{N_+, N_-} = \left[ \frac{1}{Z} \lambda_{\pm} \frac{\partial Z}{\partial \lambda_{\pm}} \right]_{\lambda_{\pm}=1}, \quad (6)$$

$$\langle N_{\pm}^2 \rangle \equiv \frac{1}{Z} \sum_{N_+=0}^{\infty} \sum_{N_-=0}^{\infty} N_{\pm}^2 Z_{N_+, N_-} = \left[ \frac{1}{Z} \lambda_{\pm} \frac{\partial}{\partial \lambda_{\pm}} \left( \lambda_{\pm} \frac{\partial Z}{\partial \lambda_{\pm}} \right) \right]_{\lambda_{\pm}=1}. \quad (7)$$

---

<sup>1</sup> e.g.  $\pi^+$  and  $\pi^-$  mesons

In thermodynamic limit,  $V \rightarrow \infty$ , and for  $Q = 0$  it gives:

$$\langle N_{\pm} \rangle = z, \quad \langle N_{\pm}^2 \rangle = z + z^2, \quad (8)$$

$$\langle N_{\pm} \rangle_{c.e.} \cong z \left( 1 - \frac{1}{4z} \right), \quad \langle N_{\pm}^2 \rangle_{c.e.} \cong z^2, \quad (9)$$

From the definition of the scaled variance (1) it then follows [7]:

$$\omega_{g.c.e.}^{\pm} \equiv \frac{\langle N_{\pm}^2 \rangle - \langle N_{\pm} \rangle^2}{\langle N_{\pm} \rangle} = 1, \quad (10)$$

$$\omega_{c.e.}^{\pm} \equiv \frac{\langle N_{\pm}^2 \rangle_{c.e.} - \langle N_{\pm} \rangle_{c.e.}^2}{\langle N_{\pm} \rangle_{c.e.}} = \frac{1}{2}. \quad (11)$$

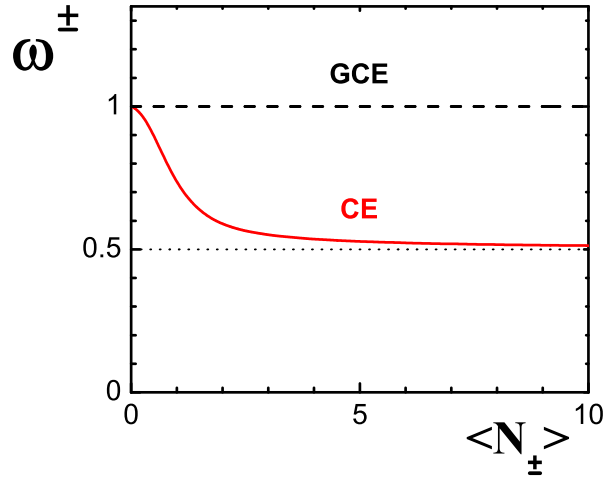


FIG. 2: The scaled variances in GCE (horizontal dashed line) and in CE (solid line) for  $Q = 0$  [7].

Thus for zero system charge in thermodynamic limit the scaled variance in CE is two times smaller than in GCE while average particle numbers are the same, see Eqs. (8) and (9) left, and Eqs. (10), (11). One can also see from Fig. 2 that the thermodynamic limit is reached very quickly. The scaled variance  $\omega_{c.e.}^{\pm}$  almost reaches its limiting value at  $\langle N_{\pm} \rangle = z \sim 5 \div 10$ .

Multiplicity fluctuations for non-zero charge in multi component system with two exactly conserved charges, namely electric charge and baryon number, are considered in [9]. The relation  $\omega_{c.e.} = \omega_{g.c.e.}/2$  is preserved in multi component system if the number of all positively and all negatively charged particles of different species is the same. Large non-zero charge  $Q > 0$  leads to additional suppression of  $\omega_{c.e.}^{+}$  and the enhancement of  $\omega_{c.e.}^{-}$ , while the relation  $\omega_{c.e.}^{+} < \omega_{c.e.}^{-} < \omega_{g.c.e.}^{\pm}$  holds. Additional baryon charge conservation leads to even stronger suppression of the scaled variance in CE comparing to GCE in thermodynamic limit.

## B. Microcanonical ensemble

The microcanonical partition function can be easily calculated analytically for the system of  $N$  noninteracting massless neutral particles if one neglects the effects of quantum statistics. This is just  $N$ -times integrated over momentum  $\delta$ -function [1]:

$$\begin{aligned} Z_N(E, V) &= \frac{1}{N!} \left( \frac{gV}{2\pi^2} \right)^N \int_0^\infty p_1^2 dp_1 \dots \int_0^\infty p_N^2 dp_N \delta \left( E - \sum_{j=1}^N p_j \right) \\ &= \frac{1}{N!} \left( \frac{gV}{\pi^2} \right)^N \frac{E^{3N-1}}{(3N-1)!} \end{aligned} \quad (12)$$

where  $E$  - is the energy and  $V$  - volume of the system. One can also generalize Eq. (12) to the system of charged particles [8]:

$$Z_{N_+, N_-}(E, V, Q) = \frac{1}{N_+! N_-!} \left( \frac{gV}{\pi^2} \right)^{N_+ + N_-} \frac{E^{3(N_+ + N_-) - 1}}{[3(N_+ + N_-) - 1]!} \delta(Q - [N_+ - N_-]), \quad (13)$$

and calculate corresponding scaled variances using Eqs. (12) and (13) similarly to (6), (7). In thermodynamic limit,  $V \rightarrow \infty$ , and for  $Q = 0$  it gives [8]:

$$\omega_{g.m.c.e.} \simeq \frac{1}{4} \left( 1 - \frac{1}{8\langle N \rangle} + \dots \right), \quad \omega_{m.c.e.}^\pm(Q=0) \simeq \frac{1}{8} \left( 1 - \frac{49}{1152\langle N_\pm^2 \rangle} + \dots \right), \quad (14)$$

where  $\langle N \rangle$  and  $\langle N_\pm \rangle$  are the average number of particles in GCE. Thus, one can see that the scaled variance in thermodynamic limit is 4 and 8 times smaller than in GCE for GMCE and MCE correspondingly, see Fig. 3.

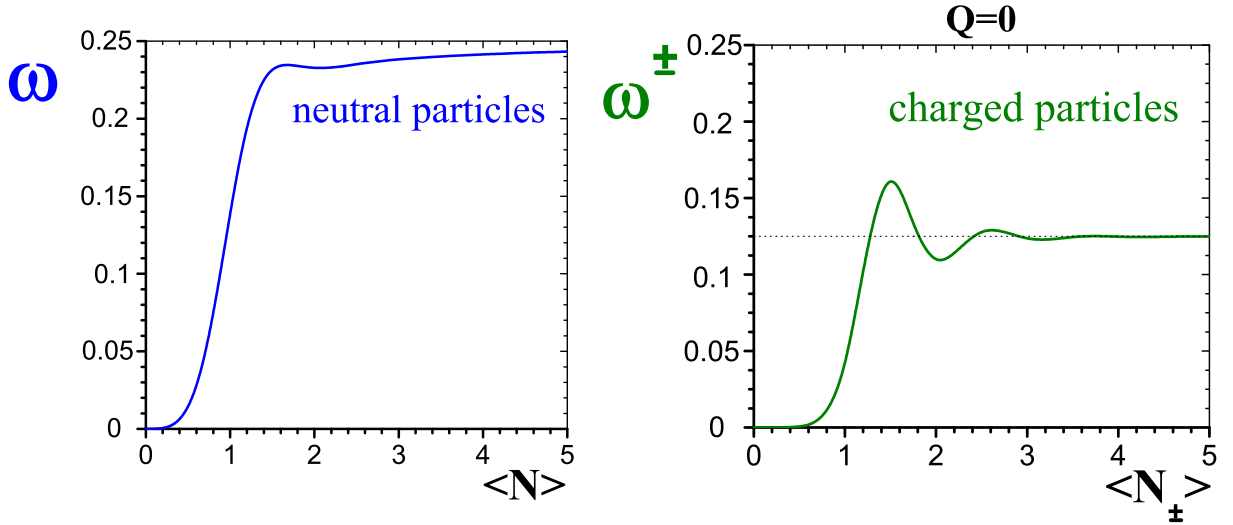


FIG. 3: The scaled variances in the GMCE, left, and in MCE, right [8].

It means that the thermodynamic equivalence for mean particle number does not apply

to fluctuations measured in terms of the scaled variance [7]-[21]:

$$\langle N \rangle \simeq \langle N \rangle_{c.e.} \simeq \langle N \rangle_{m.c.e.}, \quad V \rightarrow \infty \quad (15)$$

$$\omega_{g.c.e.} \neq \omega_{c.e.} \neq \omega_{m.c.e.}, \quad V \rightarrow \infty \quad (16)$$

see also [10] for the summary of some limiting values of the scaled variance. Note, that average particle numbers in GMCE, MCE and GCE are equivalent in thermodynamic limit [8] similarly to CE, see (8), (9), left. Canonical and microcanonical suppression [7], [8], [9] and even microcanonical enhancement [11] of average multiplicity  $\langle N \rangle$  is observed for very small systems only. Quantitatively, the limiting behavior in the MCE is reached even quicker than in CE: for  $2 \div 3$  particles if we consider  $\langle N \rangle$  or  $\langle N_{\pm} \rangle$  and for  $3 \div 4$  particles if we consider scaled variance see Fig. 3 and [8].

The analytic calculations presented above are possible only for Boltzmann statistic in CE and for Boltzmann massless particles in MCE. The inclusion of other conserved charges and quantum statistic makes the calculations technically very difficult. The simplest way to overcome these difficulties is to consider multiplicity distributions in different ensembles [12].

### III. MULTIPLICITY DISTRIBUTION

Multiplicity distribution<sup>2</sup>, partition function, different moments, variance and scaled variance are closely related, namely:

$$P(N) \equiv \frac{Z_N}{Z}, \quad \langle N^k \rangle \equiv \sum_N N^k P(N), \quad (17)$$

$$\langle (\Delta N)^2 \rangle \equiv \langle N^2 \rangle - \langle N \rangle^2, \quad \omega \equiv \frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle}. \quad (18)$$

Multiplicity distribution  $P(N)$  in ideal gas tends to Gaussian  $P_G(N)$  for  $N \gg 1$ :

$$P(N \gg 1) \simeq P_G(N) = \frac{1}{\sqrt{2\pi \omega \cdot \langle N \rangle}} \exp \left[ -\frac{(N - \langle N \rangle)^2}{2 \omega \cdot \langle N \rangle} \right], \quad (19)$$

One can easily check this for Eqs. (3), (4) and (12), (13), see the result in Fig. 4 and detailed calculations in CE [7], MCE and GMCE [8].

One can see that multiplicity distributions in different ensembles have the same maximum at  $N = \langle N \rangle$ , but different width<sup>3</sup>. As an example in Fig. 4 we choose  $N = \langle N \rangle = 10$ . One

<sup>2</sup> probability to find some number of particles  $N$  if their average number  $\langle N \rangle$  is fixed by external conditions.

<sup>3</sup> in non-relativistic case  $N = \text{const}$  by definition and  $P(N) \sim \delta(N)$  i.e. it would be a vertical line in Fig. 4.

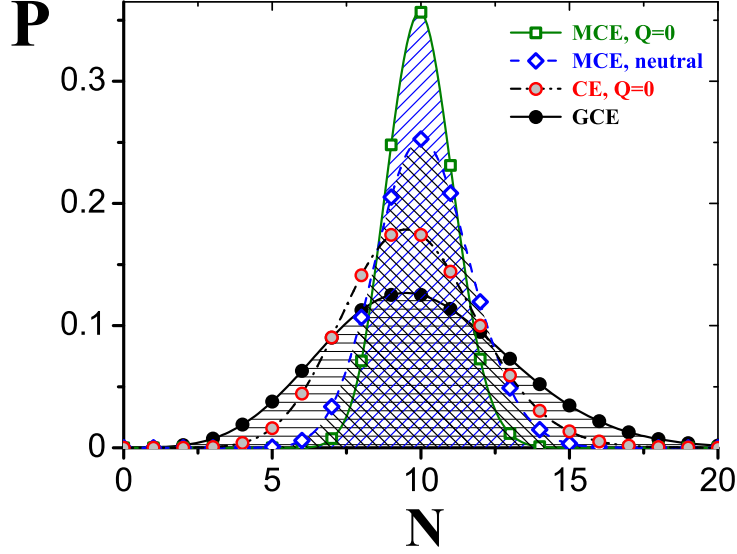


FIG. 4: Multiplicity distributions in MCE, GMCE, CE and GCE (from top to bottom) calculated by means of (13), (12), (4) and (3) correspondingly.

can also see that the distributions are smooth and have Gaussian form. Thus, quantitatively,  $N = 10$  is already big enough to consider Gaussian approximation.

To generalize our formalism for several conserved charges and include quantum statistic, let us consider again a gas of Boltzmann particles in the CE for simplicity. The GCE and CE partition function are also closely related:

$$Z(T, V, \mu_Q) = \sum_{Q=-\infty}^{\infty} e^{Q\mu_Q/T} \sum_{N_+, N_-} Z_{N_+, N_-}(T, V, Q) \quad (20)$$

The substitution of (4) in (20) transforms it to the identity [30]:

$$Z(T, V, \mu_Q) = \sum_{Q=-\infty}^{\infty} e^{Q\mu_Q/T} I_Q(2z) = \exp(2z \cosh[\mu_Q/T]) .$$

After the replacement  $e^{Q\mu_Q/T} = e^{(N_+ - N_-)\mu_Q/T}$  and  $z_{\pm} = z e^{\pm\mu_Q/T}$  one obtains:

$$\begin{aligned} Z(T, V, \mu_Q) &\equiv Z = \sum_{Q=-\infty}^{\infty} \sum_{N_+, N_-} \frac{z_+^{N_+}}{N_+!} \frac{z_-^{N_-}}{N_-!} \delta(Q - [N_+ - N_-]) \\ &= \sum_{Q=-\infty}^{\infty} \int_{-\pi}^{+\pi} \frac{d\phi}{2\pi} \exp[-iQ\phi + z(e^{\mu_Q/T+i\phi} + e^{-\mu_Q/T-i\phi})] \\ &= \sum_{Q=-\infty}^{\infty} \int_{-\pi}^{+\pi} \frac{d\phi}{2\pi} e^{-iQ\phi} \mathcal{Z}(\phi) = \sum_{Q=-\infty}^{\infty} Z_Q , \end{aligned} \quad (21)$$

where  $\mathcal{Z}(\phi)$  is the GCE partition function with replaced chemical potential  $\mu_Q/T \rightarrow \mu_Q/T + i\phi$ . Similarly to (17), the probability of finding the GCE system with the particular net-

charge  $Q$  equals the following [12]:

$$P(Q) = \frac{1}{Z} \int_{-\pi}^{+\pi} \frac{d\phi}{2\pi} e^{-iQ\phi} \mathcal{Z}(\phi) = \frac{e^{Q\mu_Q/T}}{Z} I_Q(2z). \quad (22)$$

The probability to find the number of positively charged particles  $N_+$  that is exactly equal to  $N$  in the GCE system with net-charge equal to  $Q$  is as follows [12]:

$$\begin{aligned} P(N, Q) &= \frac{1}{Z} \sum_{N_+, N_-} \frac{z_+^{N_+}}{N_+!} \frac{z_-^{N_-}}{N_-!} \delta(Q - [N_+ - N_-]) \delta(N - N_+) \\ &= \frac{1}{Z} \int_{-\pi}^{+\pi} \frac{d\phi}{2\pi} \int_{-\pi}^{+\pi} \frac{d\phi_N}{2\pi} e^{-iQ\phi} e^{-iN\phi_N} \mathcal{Z}(\phi, \phi_N) = \frac{e^{Q\mu_Q/T}}{Z} \frac{z^{2N-Q}}{N!(N-Q)!}, \end{aligned} \quad (23)$$

where  $\mathcal{Z}(\phi, \phi_N) = \exp [z (e^{\mu_Q/T + i\phi + i\phi_N} + e^{-\mu_Q/T - i\phi})]$ . Finally, the particle number distribution in CE can be found as a ratio of the distributions (23) and (22) calculated in GCE [12]:

$$P(N|Q) = \frac{P(N, Q)}{P(Q)} \quad (24)$$

One can easily check that

$$\langle N_+^k \rangle_{c.e.} = \sum_N N^k P(N|Q) = \frac{1}{I_Q(2z)} \sum_N N^k \frac{z^N}{N!} \frac{z^{N-Q}}{(N-Q)!}. \quad (25)$$

The Eq. (24) is very important, because it allows to calculate a value in CE using the values calculated in GCE. It also allows for generalization to quantum statistic and taking into account several exactly conserved charges, energy conservation, resonance decay, etc. To do this one just need to take corresponding GCE partition function and multiply it by the Fourier representations of the relevant delta functions [12]:

$$P(Q^j) = \frac{1}{Z} \left[ \prod_j \int_{-\pi}^{\pi} \frac{d\phi_j}{(2\pi)} \right] e^{-iQ^j \phi_j} \mathcal{Z}(\phi_j), \quad (26)$$

where  $j$  runs over all conserved quantities. Repeated upper and lower indexes  $j$  imply summation over  $j$ . The function  $\mathcal{Z}(\phi)$  also changes if we include different particle species and quantum statistic:

$$\mathcal{Z}(\phi_j) = \exp \left[ \sum_l z_l(\phi_j) \right], \quad (27)$$

where the single particle partition function of particle specie  $l$  is given by:

$$z_l(\phi_j) = \frac{g_l V}{(2\pi)^3} \int d^3p \ln \left[ \left( 1 \pm e^{-(\varepsilon_l - \mu_l)/T} e^{iq_l^j \phi_j} \right)^{\pm 1} \right] \equiv V \psi_l(\phi_j). \quad (28)$$

We introduced here particle  $l$ 's charges  $q_l^j = \vec{q}_l = (q_l, b_l, s_l, \dots)$  that corresponds to the charges conserved in the system. We also introduced the degeneracy factor  $g_l = (2J_l + 1)$ , internal



angular momentum  $J_l$ , mass  $m_l$ , and energy  $\varepsilon_l = \sqrt{p^2 + m_l^2}$ , the chemical potential vector  $\mu^j = (\mu_Q, \mu_B, \mu_S \dots)$ , and particle  $l$ 's chemical potential  $\mu_l = q_l^j \mu_j$ .  $V$  is the system volume, and  $T$  it's temperature. The summation  $\sum_l$  includes also anti-particles, for which  $q_l^j \rightarrow -q_l^j$ . The upper sign in the Eq. (28) denotes Fermi-Dirac statistics, while the lower is used for Bose-Einstein statistics. The Boltzmann approximation is obtained from (28) as a first term of the series expansion for  $e^{-(\varepsilon_l - \mu_l)/T} \ll 1$ .

The real calculations of (26) can be performed only in the limit  $V \rightarrow \infty$ . Then the main contribution to the integral in (26) comes from a small region around the origin. Thus it is possible to make the Taylor expansion of  $\sum_l \psi_l$  and leave only the first two terms. Similar saddle point expansion was intensively used for partition function itself [17], [21], [22], [23] while the relations between partition function, multiplicity distribution, and scaled variance was obtained only in [12].

It was shown for GMCE in [13] and for the most general case of MCE with arbitrary number of conserved charges in [12] that the variance is proportional to the ratio of correlation matrix determinants:

$$\langle (\Delta N)^2 \rangle = V \frac{\det |\tilde{A}|}{\det |A|} \quad (29)$$

where the elements of the correlation matrixes can be found as follows:

$$A_{i,j} = - \frac{\partial^2 \log \mathcal{Z}(\phi_j)}{\partial \phi_i \partial \phi_j} \Big|_{\vec{\phi}=0}, \quad \tilde{A}_{i,j} = \frac{\partial^2 \log \mathcal{Z}(\phi_j, \phi_N)}{\partial \phi_i \partial \phi_j} \Big|_{\vec{\phi}, \phi_N=0}. \quad (30)$$

The difference between  $A$  and  $\tilde{A}$  is that in the latter case  $i$  and  $j$  run over  $N$  also [12]. Then the scaled variance is a ratio of (30) to the mean multiplicity:

$$\langle N \rangle = -i \frac{\partial \log \mathcal{Z}(\phi_j, \phi_N)}{\partial \phi_N} \Big|_{\vec{\phi}, \phi_N=0}. \quad (31)$$

The above method is very powerful. Nevertheless it fails in the case of Bose condensation, because scaled variance in GCE then goes to infinity [14] and multiplicity distribution has infinite width. All matrix elements (30) and higher derivatives of  $\mathcal{Z}(\phi)$  tends to infinity in GCE [12]. However, exact charge and energy conservation suppress even these infinite fluctuations [14], [15]. The very special selection of events is need to see these infinite fluctuations in MCE. This is proposed as the signal of possible  $\pi$ -meson condensation in  $p + p$  collisions [15], [16].

The further improvement is possible if one consider average multiplicity and fluctuations at different momentum levels. This approach is called the microcorrelator method [8]. It

analogous to the above approach [12], but additionally allows to consider correlations between different momentum levels. The full hadron gas in the next section is considered using microcorrelator method [17], [20].

#### IV. HADRON GAS

Let us consider the fluctuations in the ideal relativistic gas with different types of hadrons in the MCE with exactly fixed the global electric (Q), baryon (B), and strange (S) charges of the statistical system. The system of non-interacting Bose or Fermi particles of species  $i$  can be characterized by the occupation numbers  $n_{p,i}$  of single quantum states labelled by momenta  $p$ . The occupation numbers run over  $n_{p,i} = 0, 1$  for fermions and  $n_{p,i} = 0, 1, 2, \dots$  for bosons. The GCE average values and fluctuations of  $n_{p,i}$  equal the following [29]:

$$\langle n_{p,i} \rangle = \frac{1}{\exp \left[ \left( \sqrt{p^2 + m_i^2} - \mu_i \right) / T \right] - \gamma_i}, \quad (32)$$

$$v_{p,i}^2 \equiv \langle \Delta n_{p,i}^2 \rangle \equiv \langle (n_{p,i} - \langle n_{p,i} \rangle)^2 \rangle = \langle n_{p,i} \rangle (1 + \gamma_i \langle n_{p,i} \rangle). \quad (33)$$

In Eq. (32),  $T$  is the system temperature,  $m_i$  is the mass of  $i$ -th particle species,  $\gamma_i$  corresponds to different statistics (+1 and  $-1$  for Bose and Fermi, respectively, and  $\gamma_i = 0$  gives the Boltzmann approximation), and chemical potential  $\mu_i$  equals:

$$\mu_i = q_i \mu_Q + b_i \mu_B + s_i \mu_S, \quad (34)$$

where  $q_i$ ,  $b_i$ ,  $s_i$  are the electric charge, baryon number and strangeness of particle of specie  $i$ , respectively, while  $\mu_Q$ ,  $\mu_B$ ,  $\mu_S$  are the corresponding chemical potentials which regulate the average values of these global conserved charges in the GCE.

The average number of particles of species  $i$ , the number of positively and negatively charged particles are equal:

$$\langle N_i \rangle \equiv \sum_p \langle n_{p,i} \rangle = \frac{g_i V}{2\pi^2} \int_0^\infty p^2 dp \langle n_{p,i} \rangle, \quad \langle N_+ \rangle = \sum_{i, q_i > 0} \langle N_i \rangle, \quad \langle N_- \rangle = \sum_{i, q_i < 0} \langle N_i \rangle, \quad (35)$$

where  $g_i$  is the degeneracy factor of particle of species  $i$ . A sum of the momentum states means the momentum integral, which holds in the thermodynamic limit  $V \rightarrow \infty$ .

Particle number fluctuations and correlations can be calculated in all ensembles using the microscopic correlator method.

$$\langle \Delta N_i \Delta N_j \rangle \dots = \sum_{p,k} \langle \Delta n_{p,i} \Delta n_{k,j} \rangle \dots, \quad (36)$$

where  $\langle \dots \rangle$ ... means GCE, CE, or MCE microscopic correlator. The scaled variances of negatively and positively charged particles read:

$$\omega^- = \frac{\langle (\Delta N_-)^2 \rangle}{\langle N_- \rangle}, \quad \omega^+ = \frac{\langle (\Delta N_+)^2 \rangle}{\langle N_+ \rangle}, \quad (37)$$

where

$$\langle (\Delta N_-)^2 \rangle = \sum_{i,j; q_i < 0, q_j < 0} \langle \Delta N_i \Delta N_j \rangle, \quad \langle (\Delta N_+)^2 \rangle = \sum_{i,j; q_i > 0, q_j > 0} \langle \Delta N_i \Delta N_j \rangle. \quad (38)$$

The microscopic correlator in the GCE reads:

$$\langle \Delta n_{p,i} \Delta n_{k,j} \rangle = v_{p,i}^2 \delta_{ij} \delta_{pk}, \quad (39)$$

where  $v_{p,i}^2$  is given by Eq. (33). This gives a possibility to calculate the fluctuations of different observables in the GCE. Note that only particles of the same species,  $i = j$ , and on the same level,  $p = k$ , do correlate in the GCE. Thus, Eq. (39) is equivalent to Eq. (33): only the Bose and Fermi effects for the fluctuations of identical particles on the same level are relevant in the GCE.

The MCE microscopic correlator is as follows [17], [20]:

$$\begin{aligned} \langle \Delta n_{p,i} \Delta n_{k,j} \rangle_{m.c.e.} &= v_{p,i}^2 \delta_{ij} \delta_{pk} - \frac{v_{p,i}^2 v_{k,j}^2}{|A|} [ q_i q_j M_{qq} + b_i b_j M_{bb} + s_i s_j M_{ss} \\ &+ (q_i s_j + q_j s_i) M_{qs} - (q_i b_j + q_j b_i) M_{qb} - (b_i s_j + b_j s_i) M_{bs} \\ &+ \epsilon_{pi} \epsilon_{kj} M_{\epsilon\epsilon} - (q_i \epsilon_{pj} + q_j \epsilon_{ki}) M_{q\epsilon} + (b_i \epsilon_{pj} + b_j \epsilon_{ki}) M_{b\epsilon} - (s_i \epsilon_{pj} + s_j \epsilon_{ki}) M_{s\epsilon} ], \end{aligned} \quad (40)$$

where  $|A|$  is the determinant and  $M_{ij}$  are the minors of the following matrix:

$$A = \begin{pmatrix} \Delta(q^2) & \Delta(bq) & \Delta(sq) & \Delta(\epsilon q) \\ \Delta(qb) & \Delta(b^2) & \Delta(sb) & \Delta(\epsilon b) \\ \Delta(qs) & \Delta(bs) & \Delta(s^2) & \Delta(\epsilon s) \\ \Delta(q\epsilon) & \Delta(b\epsilon) & \Delta(s\epsilon) & \Delta(\epsilon^2) \end{pmatrix}, \quad (41)$$

with the elements,  $\Delta(q^2) \equiv \sum_{p,k} q_k^2 v_{p,k}^2$ ,  $\Delta(qb) \equiv \sum_{p,k} q_k b_k v_{p,k}^2$ ,  $\Delta(q\epsilon) \equiv \sum_{p,k} q_k \epsilon_{pk} v_{p,k}^2$ , etc. The sum,  $\sum_{p,k}$ , means integration over momentum  $p$ , and summation over all hadron-resonance species  $k$  contained in the model. Note that the presence of MCE terms containing single particle energies,  $\epsilon_{pi} = \sqrt{p^2 + m_i^2}$ , in the last line of Eq.(40) is a consequence of exact energy conservation. In the CE, only charges are conserved exactly, thus the terms of the last line in Eq. (40) are absent, and  $A$  in Eq. (41) becomes the  $3 \times 3$  matrix (see Ref. [17]).

## V. EFFECT OF RESONANCE DECAYS

The average number of  $i$ -particles in the presence of primary particles  $N_i^*$  and different resonance types  $R$  is the following:

$$\langle N_i \rangle = \langle N_i^* \rangle + \sum_R \langle N_R \rangle \sum_r b_r^R n_{i,r}^R \equiv \langle N_i^* \rangle + \sum_R \langle N_R \rangle \langle n_i \rangle_R \quad (42)$$

The summation  $\sum_R$  runs over all types of resonances. The  $\langle \dots \rangle$  and  $\langle \dots \rangle_R$  correspond to the GCE averaging, and that over resonance decay channels. Resonance decay has a probabilistic character. This itself causes the particle number fluctuations in the final state. In the GCE the final state correlators can be calculated as [24]:

$$\langle \Delta N_i \Delta N_j \rangle = \langle \Delta N_i^* \Delta N_j^* \rangle + \sum_R [\langle \Delta N_R^2 \rangle \langle n_i \rangle_R \langle n_j \rangle_R + \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R] , \quad (43)$$

where  $b_r^R$  is the branching ratio of the  $r$ -th branch,  $n_{i,r}^R$  is the number of  $i$ -th particles produced in that decay mode, and  $r$  runs over all branches with the requirement  $\sum_r b_r^R = 1$  and  $\langle \Delta n_i \Delta n_j \rangle_R \equiv \sum_r b_r^R n_{i,r}^R n_{j,r}^R - \langle n_i \rangle_R \langle n_j \rangle_R$ . Note that different branches are defined in a way that final states with only stable (with respect to strong and electromagnetic decays) hadrons are counted.

All primary particles and resonances become to correlate in the presence of exact charge conservation laws. Thus for the MCE correlators we obtain a new result [17]:

$$\begin{aligned} \langle \Delta N_i \Delta N_j \rangle_{m.c.e.} &= \langle \Delta N_i^* \Delta N_j^* \rangle_{m.c.e.} + \sum_R \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R + \sum_R \langle \Delta N_i^* \Delta N_R \rangle_{m.c.e.} \langle n_j \rangle_R \\ &+ \sum_R \langle \Delta N_j^* \Delta N_R \rangle_{m.c.e.} \langle n_i \rangle_R + \sum_{R,R'} \langle \Delta N_R \Delta N_{R'} \rangle_{m.c.e.} \langle n_i \rangle_R \langle n_j \rangle_{R'} . \end{aligned} \quad (44)$$

Additional terms in Eq. (44) compared to Eq. (43) are due to the correlations induced by exact charge conservations in the MCE. The Eq. (44) remains valid in the CE too with  $\langle \dots \rangle_{m.c.e.}$  replaced by  $\langle \dots \rangle_{c.e.}$ , the difference between them appears only when one specifies the microscopic correlators (40) of the MCE or CE.

## VI. SCALED VARIANCES ALONG THE CHEMICAL FREEZE-OUT LINE

Mean hadron multiplicities in heavy ion collisions at high energies can be approximately fitted by the GCE hadron-resonance gas model. The fit parameters are temperature  $T$ , chemical potentials  $(\mu_B, \mu_S, \mu_Q)$ , and strangeness suppression factor  $\gamma_S$ , which allows for non-equilibrium strange hadron yields. There are several programs designed for the analysis

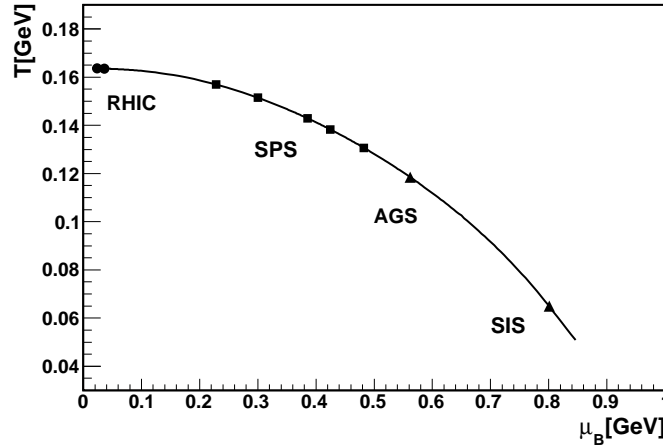


FIG. 5: The chemical freeze-out line in central A+A collisions [17].

of particle multiplicities in relativistic heavy-ion collisions within the hadron-resonance gas model, see e.g., SHARE [25], THERMUS [26] and THERMINATOR [27]. In this paper an extended version of the THERMUS thermal model framework [26] is used.

For the chemical freeze-out condition we choose the average energy per particle  $\langle E \rangle / \langle N \rangle = 1 \text{ GeV}$  [28]. Using the standard parametrization [5] we obtain the  $T - \mu_B$  freeze-out line for central A+A collisions (see Fig. 5). The center of mass nucleon-nucleon energies,  $\sqrt{S_{NN}}$ , marked in the figures below correspond to the beam energies at SIS (2A GeV), AGS (11.6A GeV), SPS (20A, 30A, 40A, 80A, and 158A GeV), colliding energies at RHIC ( $\sqrt{S_{NN}} = 62.4 \text{ GeV}$ , 130 GeV and 200 GeV) and LHC ( $\sqrt{S_{NN}} = 5500 \text{ GeV}$ ).

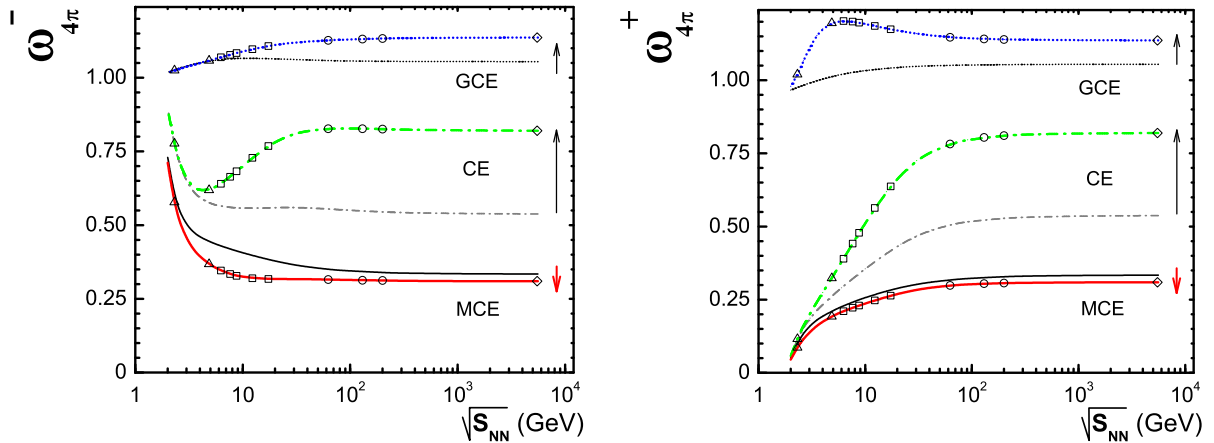


FIG. 6: The scaled variances for negatively and positively charged particles, both primordial and final, along the chemical freeze-out line for central Pb+Pb (Au+Au) collisions. Different lines present the GCE, CE, and MCE results. Symbols at the CE and MCE lines for the final particles correspond to the specific collision energies. The arrows show the effect of resonance decays [20].

Figure 6 show the prediction for the scaled variances for negatively and positively charged particles as a function of  $\sqrt{s_{NN}}$ .

The prediction can be compared with the preliminary NA49 data on Pb+Pb collisions at 20A-158A GeV [19] using the following approximate formula:

$$\omega_{acc}^{\pm} = 1 - q + q \omega_{4\pi}^{\pm}, \quad (45)$$

where  $\omega_{4\pi}$  refers to an ideal detector with full  $4\pi$ -acceptance and  $\omega_{acc}^{\pm}$  is the scaled variance measured by a real detector with a limited acceptance),  $q$  is the ratio between mean multiplicities of accepted particles and all hadrons. In the limit of a very ‘bad’ (or ‘small’) detector,  $q \rightarrow 0$ , all scaled variances approach linearly to 1, i.e., this would lead to the Piossonian distributions for detected particles. However, we find a strong qualitative difference between the predictions of the statistical model valid for any freeze-out conditions and experimental acceptances: the CE and MCE correspond to  $\omega_{m.c.e.}^{\pm} < \omega_{c.e.}^{\pm} < 1$ , and the GCE to  $\omega_{g.c.e.}^{\pm} > 1$ .

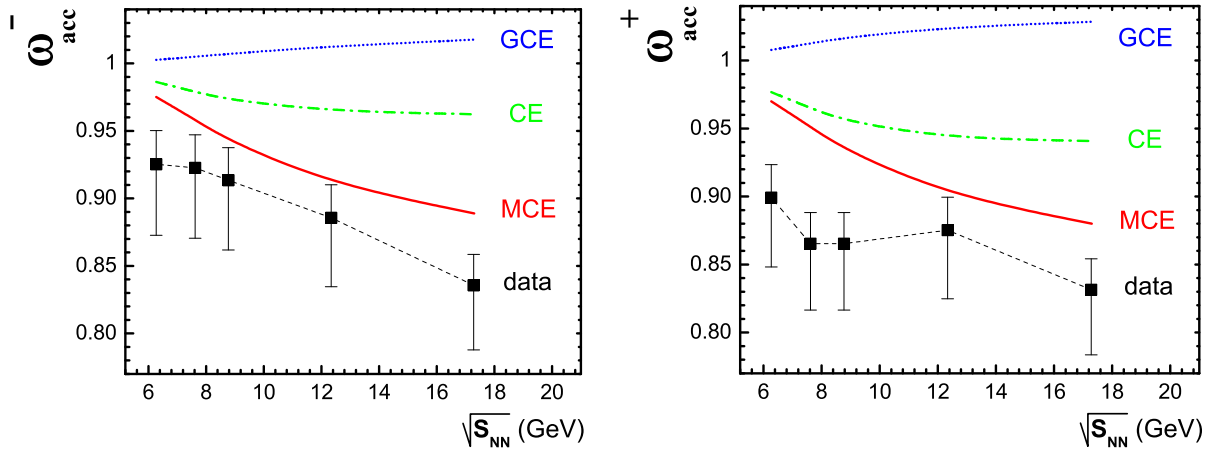


FIG. 7: The scaled variances for negative (left) and positive (right) hadrons along the chemical freeze-out line for central Pb+Pb collisions at the SPS energies. The corresponding  $T$  and  $\mu_B$  values at different SPS collision energies are presented in Fig. 5. Different lines show the GCE, CE, and MCE results calculated with the NA49 experimental acceptance [20].

From Fig. 7 it follows that the NA49 data for  $\omega^{\pm}$  extracted from the most central Pb+Pb collisions at all SPS energies are close to the results of the hadron-resonance gas statistical model within the MCE. The data reveal even stronger suppression of the particle number fluctuations. A possible reason of this is an uncertainty in the determination of the detector acceptance and an additional suppression due to momentum conservation and the excluded volume effects in the hadron-resonance gas.

In order to allow for a detailed comparison of the distributions the ratio of the data and the model distributions to the Poisson one is presented in Fig. 8. The convex shape of

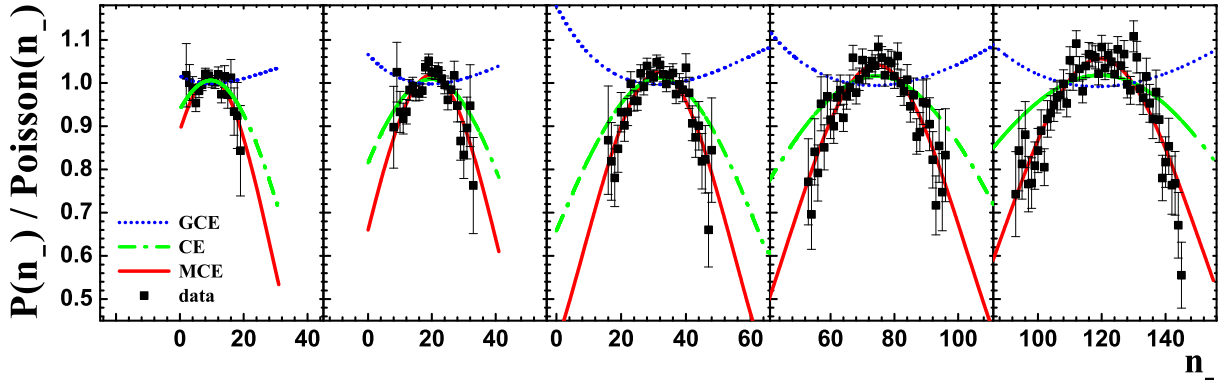


FIG. 8: The ratio of the multiplicity distributions to Poisson ones for negatively charged hadrons produced in central (1%) Pb+Pb collisions at 20A GeV, 30A GeV, 40A GeV, 80A GeV, and 158A GeV (from left to right) in the NA49 acceptance [19]. The preliminary experimental data (solid points) of NA49 [19] are compared with the prediction of the hadron-resonance gas model obtained within different statistical ensembles, the GCE (dotted lines), the CE (dashed-dotted lines), and the MCE (solid lines) [20].

the data reflects the fact that the measured distribution is significantly narrower than the Poisson one. This suppression of fluctuations is observed at all five SPS energies and it is consistent with the results for the scaled variance shown and discussed previously. The GCE hadron-resonance gas results are broader than the corresponding Poisson distribution. The ratio has a concave shape. An introduction of the quantum number conservation laws (the CE results) leads to the convex shape and significantly improves agreement with the data. Further improvement of the agreement is obtained by the additional introduction of the energy conservation law (the MCE results). The measured spectra surprisingly well agree with the MCE predictions [20].

## VII. SUMMARY

We have found that scaled variances are different in different statistical ensembles. For relativistic one component Boltzmann gas with zero charge in thermodynamic limit we analytically obtained rather interesting limiting values:  $\omega_{g.c.e.} = 1$ ,  $\omega_{c.e.} = 1/2$ ,  $\omega_{g.m.c.e.}(m = 0) = 1/4$  and  $\omega_{m.c.e.}(m = 0) = 1/8$ . We also found an analytical method to account for

resonance decays. The formalism that allows to consider any number of conserved charges and also energy conservation in full hadron-resonance gas was developed.

The experimental data allows to exclude GCE for scaled variance. They show reasonable agreement with CE and surprisingly well agree with the expectations for the MCE. Thus the predicted suppression of the multiplicity fluctuations in relativistic gases in the thermodynamic limit due to conservation laws do exist.

### Acknowledgments

I would like to thank my PhD supervisor Mark I. Gorenstein and also my co-authors: F. Becattini, L. Ferroni, M. Gazdzicki, M. Hauer, A. Keranen, V. P. Konchakovski, A. P. Kostyuk and O. S. Zozulya. I would like also to thank for the support The International Association for the Promotion of Cooperation with Scientists from the New Independent states of the Former Soviet Union (INTAS), Ref. Nr. 06-1000014-6454.

- 
- [1] E. Fermi, Prog. Theor. Phys. **5**, 570 (1950).
  - [2] L.D. Landau, Izv. Akad. Nauk SSSR, Ser. Fiz. **17**, 51 (1953).
  - [3] R. Hagedorn, Nucl. Phys. B **24**, 93 (1970).
  - [4] P. Braun-Munzinger, K. Redlich, and J. Stachel, Review for Quark Gluon Plasma 3, eds. R.C. Hwa and X.-N. Wang, World Scientific, Singapore, 2004, [nucl-th/0304013].
  - [5] J. Cleymans, H. Oeschler, K. Redlich, and S. Wheaton, Phys. Rev. C **73**, 034905 (2006); F. Becattini, J. Manninen, and M. Gazdzicki, Phys. Rev. C **73**, 044905 (2006).
  - [6] H. Heiselberg, Phys. Rep. **351**, 161 (2001); S. Jeon and V. Koch, Review for Quark-Gluon Plasma 3, eds. R.C. Hwa and X.-N. Wang, World Scientific, Singapore, 430-490 (2004), [hep-ph/0304012].
  - [7] V.V. Begun, M. Gazdzicki, M.I. Gorenstein, and O.S. Zozulya, Phys. Rev. C **70**, 034901 (2004).
  - [8] V.V. Begun, M.I. Gorenstein, A.P. Kostyuk and O.S. Zozulya, Phys. Rev. C **71** (2005) 054904
  - [9] V.V. Begun, M.I. Gorenstein and O.S. Zozulya, Phys. Rev. C **72** (2005) 014902
  - [10] V.V. Begun and M.I. Gorenstein, *Prepared for International Conference on New Trends in High Energy Physics (Experiment, Phenomenology, Theory), Yalta, Crimea, Ukraine, 10- 17 Sep 2005*



- [11] V.V. Begun, L. Ferroni, M.I. Gorenstein, M. Gazdzicki and F. Becattini, J. Phys. G **32** (2006) 1003
- [12] M. Hauer, V.V. Begun and M.I. Gorenstein, arXiv:0706.3290 [nucl-th].
- [13] V.V. Begun, M.I. Gorenstein, A.P. Kostyuk and O.S. Zozulya, J. Phys. G **32** (2006) 935
- [14] V.V. Begun and M.I. Gorenstein, Phys. Rev. C **73** (2006) 054904
- [15] V.V. Begun and M.I. Gorenstein, Phys. Lett. B **653** (2007) 190
- [16] V.V. Begun and M.I. Gorenstein, arXiv:0709.1434 [hep-ph].
- [17] V.V. Begun, M.I. Gorenstein, M. Hauer, V.P. Konchakovski and O.S. Zozulya, Phys. Rev. C **74** (2006) 044903
- [18] J. Cleymans, K. Redlich, and L. Turko, Phys. Rev. C **71**, 047902 (2005); J. Phys. G **31**, 1421 (2005);
- [19] B. Lungwitz et al., [for the NA49 Collaboration], in proceedings of *Correlations and Fluctuations in Relativistic Nuclear Collisions*, July 7-9, 2006, Florence, Italy. B. Lungwitz et al., PoS C **FRNC2006**, 024 (2006)
- [20] V.V. Begun, M. Gazdzicki, M.I. Gorenstein, M. Hauer, V.P. Konchakovski and B. Lungwitz, Phys. Rev. C **76** (2007) 024902
- [21] A. Keranen, F. Becattini, V.V. Begun, M.I. Gorenstein and O.S. Zozulya, J. Phys. G **31** (2005) S1095
- [22] F. Becattini, A. Keranen, L. Ferroni and T. Gabbriellini, Phys. Rev. C **72** (2005) 064904
- [23] F. Becattini, Z.Phys., C **69**, (1996); F. Becattini, U. Heinz, ibid. **76**, (1997);
- [24] S. Jeon and V. Koch, Phys. Rev. Lett. **83**, 5435 (1999).
- [25] G. Torrieri, S. Steinke, W. Broniowski, W. Florkowski, J. Letessier and J. Rafelski, Comput. Phys. Commun. **167** (2005) 229
- [26] S. Wheaton, J. Cleymans, J. Phys. G **31** 1069-1074 (2005)
- [27] A. Kisiel, T. Taluc, W. Broniowski and W. Florkowski, Comput. Phys. Commun. **174**, 669 (2006)
- [28] J. Cleymans and K. Redlich, Phys. Rev. Lett. **81**, 5284 (1998).
- [29] L.D. Landau and E.M. Lifshitz. *Statistical Physics (Course of Theoretical Physics, Volume 5)*. Pergamon Press Ltd. 1980.
- [30] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York, Dover (1965).